

A quantum group for the Einstein equations

Giuseppe Iurato*

Abstract

In this paper, we expose the construction of a possible, simple matrix quantum group structure (according to Woronowicz), related to elementary formal aspects of the Einstein field equations of General Relativity, and its possible symmetries.

Mainly, we present a simple application of the results achieved by M. Dubois-Violette and G. Launer in [1], where is built up a first matrix quantum group structure (in the sense of S.L. Woronowicz – see [8], 2.1) associated to an arbitrary non-degenerate bilinear form.

Precisely, we apply, almost verbatim, these considerations to a generalization of the *Einstein field equations* (1915), in purely covariant form given by¹ (see [6], 1.13.5, and [5], 4.0, 4.1)

$$(1) \quad G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi GT_{ij}, \quad i, j = 0, 1, 2, 3,$$

where G_{ij} is the *Einstein curvature tensor*, R_{ij} is the *Ricci curvature tensor*, g_{ij} is the *Lorentz metric*, R is the *Ricci scalar*, G is the *gravitational constant*, and T_{ij} is the so called *Hilbert tensor* (see [9], Chapter 7) with $T_{ij} = T_{ij}(g_{lh}, \partial g_{lh}; \psi_k, \partial \psi_k)$ in the presence of a set of physical fields ψ_k $k = 1, \dots, p$. In the geometrized units, it is $G = 1$ (see [7]).

We recall that the Einstein field equations (1) may be deduced both by a variational Palatini's argument (see [9]) and, inductively, by the newtonian Poisson's equation $\Delta\phi = 4\pi G\rho$.

Following the latter way, it is assumed that the field equations for the gravitational field, that we may call *generalized Einstein (field) equations*, should

*e-mail: iurato@dmf.unict.it

¹According to the Robertson-Noonan sign convention (1968) (see [4]).

have a general form of the type (see [9], Chapter 4, [10], Cap. II, § 2.1, and [4], Chapter 17, § 17.1)

$$(2) \quad G_{ij}(g_{lh}, g_{lh,r}, g_{lh,rt}, \dots) = k\pi T_{ij}, \quad i, j = 0, 1, 2, 3,$$

where G_{ij} is a yet to be determined tensor function of the metric tensor g_{lh} and some of its derivatives, and k is a real constant.

Many physical reasons (see [10], § 2.1, and [4], § 17.1) restricts the class of the possible functions G_{ij} , satisfying (2), to a well-defined tensor, namely the Einstein curvature tensor mentioned above, obtaining the known equations (1).

On the other hand, by earlier Weyl's and Cartan's results culminated in Lovelock's statement (see [16]), if we seek a tensor equation of the form $G_{ij} = T_{ij}$, where the components A_{ij} involve the metric tensor g_{ij} and its first and second derivatives (hence, assuring second-order partial differential equations generalizing the Poisson one), and if A_{ij} have vanishing divergence $A_{ij;j} = 0$, then the equation must be of the form $aG_{ij} + bg_{ij} = -8\pi kT_{ij}$, where a and b are constants; Einstein's choice is then $a = 1, b = 0$ (b is said to be the *cosmological constant*).

At this point, taking into account the geometrical meaning of the Einstein's equations² according to [17], it is possible to consider the following *Einstein bilinear form*

$$(2') \quad \Omega_{ij} \doteq G_{ij} + 8\pi T_{ij}, \quad i, j = 0, 1, 2, 3,$$

that, for now, we suppose to be non-degenerate; its zero values are the generalized Einstein equations (2). In (2'), we suppose, a priori, G_{ij} to be an arbitrary bilinear form (of \mathbb{R}^4), while T_{ij} is the Hilbert tensor.

In [1] (see also [11], Example 4.62), it is considered a finite family $\{T(\alpha)\}_{\alpha \in \Xi}$ of (r_α, s_α) -tensors on \mathbb{R}^n and the group G of the automorphisms of \mathbb{R}^n that preserve $T(\alpha)$ in the following sense

$$(3) \quad u_{k_1}^{i_1} \dots u_{k_{r_\alpha}}^{i_{r_\alpha}} T(\alpha)_{j_1 \dots j_{s_\alpha}}^{k_1 \dots k_{r_\alpha}} = u_{j_1}^{k_1} \dots u_{j_{s_\alpha}}^{k_{s_\alpha}} T(\alpha)_{k_1 \dots k_{s_\alpha}}^{i_1 \dots i_{r_\alpha}} \quad \forall \alpha \in \Xi,$$

supposing invertible the generic matrix $u = \|u_j^i\| \in G$.

In matrix quantum group theory (see [2]), one can considers the elements

²These arguments shall be the matter of another paper.

u_j^i as linear coordinate functions on G , which assigns to each $g \in G$ its matrix elements (respect to a given base), namely $u_j^i(g) = g_j^i$, and that one can also interpret as generating the unital associative algebra $Fun(G)$ of functions on G , under the relations (3).

The latter is a commutative Hopf algebra, with usual comultiplication given by $(\Delta(f))(g_1, g_2) = f(g_1 g_2)$, so that the cocommutativity, or not, of this algebra, is related to the commutativity, or not, of the group G ; furthermore, the coproduct is induced by $\Delta u_j^i = u_k^i \otimes u_j^k$, since $u_j^i(g) = g_j^i$.

Hence, following [1], we could say that (3) defines a first (matrix) quantum group structure preserving each $T(\alpha)$; moreover, we restricts our study to the case in which $T(\alpha)$ is a given non-degenerate bilinear form Ω_{ij} on \mathbb{R}^4 , with dual Ω^{ij} (given by the inverse matrix), that is we suppose $r_\alpha = 0$, $s_\alpha = 2$, $card \Xi = 1$ and³ $n = 4$.

If Ω is a bilinear form on \mathbb{R}^4 with components (respect to a given base) Ω_{ij} , and $\tilde{\Omega}$ is a bilinear form on its dual with components (respect to the dual base) $\tilde{\Omega}^{ij}$, then, as known, $\tilde{\Omega} \otimes \Omega$ is identified with the endomorphisms of $\mathbb{R}^4 \otimes \mathbb{R}^4$ with components $\Omega^{i_1 i_2} \Omega_{j_1 j_2}$; likewise, if u and v are endomorphisms of \mathbb{R}^4 with components u_j^i and v_j^i , then $u \otimes v$ is identified with the endomorphism of $\mathbb{R}^4 \otimes \mathbb{R}^4$ with components $u_{j_1}^{i_1} v_{j_2}^{i_2}$.

Let Ω be the non-degenerate bilinear form with components (in the canonical base) Ω_{ij} given by (3); the matrix of its components Ω_{ij} , will be denoted again by Ω . Associated to Ω is its dual Ω^{-1} of $\mathbb{R}^4 \otimes \mathbb{R}^4$, that is the bilinear form on the dual of \mathbb{R}^4 ($\cong \mathbb{R}^4$), with components Ω^{ij} defined by $\Omega^{ik} \Omega_{kj} = \delta_j^i$; the matrix of the components Ω^{ij} will be again denoted by Ω^{-1} , the inverse of the matrix Ω (that there exists because Ω is non-degenerate).

Let $\mathcal{A}_{\mathbb{R}}(\Omega)$ be the unital associative \mathbb{R} -algebra generated by the scalars $t_j^i \in \mathbb{R}$ $i, j = 0, 1, 2, 3$, with the relations

$$\Omega_{ij} t_k^i t_l^j = \Omega_{kl}, \quad \Omega^{ij} t_i^k t_j^l = \Omega^{kl}, \quad k, l = 0, 1, 2, 3,$$

where $\Omega_{kl}, \Omega^{kl} \in \mathbb{R}$ are identified, respectively, with $\Omega_{kl} 1_{\mathcal{A}}, \Omega^{kl} 1_{\mathcal{A}} \in \mathcal{A}_{\mathbb{R}}(\Omega)$, if $1_{\mathcal{A}}$ is the unit of $\mathcal{A}_{\mathbb{R}}(\Omega)$.

Hence, it is possible to prove (see [1]) that

1. there exists a unique homomorphism of algebras, say $\Delta : \mathcal{A}_{\mathbb{R}}(\Omega) \rightarrow \mathcal{A}_{\mathbb{R}}(\Omega) \otimes \mathcal{A}_{\mathbb{R}}(\Omega)$, such that $\Delta t_j^i = t_k^i \otimes t_j^k$ $i, j = 0, 1, 2, 3$;

³However, the following considerations holds true also for any $n \geq 2$.

2. there exists a unique homomorphism of algebras, say $\varepsilon : \mathcal{A}_{\mathbb{R}}(\Omega) \rightarrow \mathbb{R}$, such that $\varepsilon(t_j^i) = \delta_j^i$ $i, j = 0, 1, 2, 3$;
3. there exists a unique linear antimultiplicative mapping, say $S : \mathcal{A}_{\mathbb{R}}(\Omega) \rightarrow \mathcal{A}_{\mathbb{R}}(\Omega)$, such that⁴ $S(t_j^i) = \Omega^{ik} t_k^l \Omega_{lj}$ $i, j = 0, 1, 2, 3$, and $S(1_{\mathcal{A}}) = 1_{\mathcal{A}}$.

Furthermore, Δ is a coproduct, ε is a counit, and S is an antipode⁵ since $S(t_k^i) t_j^k = t_k^i S(t_j^k)$, so that, denoted by m the product of the algebra $\mathcal{A}_{\mathbb{R}}(\Omega)$, we have that $(\mathcal{A}_{\mathbb{R}}(\Omega), m, 1_{\mathcal{A}}, \Delta, \varepsilon, S)$ is a Hopf algebra, called the *Hopf algebra of the Einstein bilinear form* Ω .

This Hopf algebra defines (in the terminology of [1]; see also [12], Appendix 2) the quantum group of the non-degenerate bilinear form Ω , that we may call the *Einstein quantum group*; hence, as usual, we may think $\mathcal{A}_{\mathbb{R}}(\Omega)$ as a kind of algebra of 'representative functions' on this quantum group.

Besides, this quantum group extends the classical group of the linear transformations of \mathbb{R}^4 which preserves Ω , and, therefore, such a quantum object may represents further generalized symmetries of the Einstein bilinear form Ω .

The matrix $t = \|t_j^i\| \in M^{(4,4)}(\mathcal{A}_{\mathbb{R}}(\Omega))$ is a multiplicative matrix (see [3]) whose entries generates $\mathcal{A}_{\mathbb{R}}(\Omega)$, obtaining an example of matrix quantum group.

Note. All the above considerations about $\mathcal{A}_{\mathbb{R}}(\Omega)$, holds for an arbitrary non-degenerate bilinear form Ω of \mathbb{R}^n , with $n \geq 2$.

Given a non-degenerate bilinear form Ω on \mathbb{R}^n ($n \geq 2$), with components Ω_{ij} (respect to the canonical base), we may define the quadratic homogeneous algebra⁶ $\mathcal{Q}_{\mathbb{R}}(\Omega)$ generated by the elements x^j $j = 1, \dots, n$, with the relations $\Omega_{ij} x^i x^j = 0$.

In [12], § 2. (see also [13]), it is proved as $\mathcal{Q}_{\mathbb{R}}(\Omega)$ be a Gorenstein and Koszul algebra of global dimension 2. Conversely, it is possible to prove that any quadratic algebra generated by n elements x^j , finitely generated in degree 1 and finitely presented with relations of degree ≥ 2 , which is Gorenstein and Koszul of low global dimension 2, is an algebra of the type $\mathcal{Q}_{\mathbb{R}}(\Omega)$ for a

⁴Setting $t = \|t_j^i\|$, it is $S(t) = (\Omega^{-1})^t t \Omega$.

⁵In general, there is no antipode for a generic tensor $T(\alpha)$.

⁶For brief recalls on homogeneous algebras, see [13] or [12], Appendix 1, and references therein.

certain non-degenerate bilinear form Ω .

Moreover, if $\Omega \xrightarrow{\chi} \Omega \circ M$ is the action given by $(\Omega \circ M)(x, y) = \Omega(Mx, My)$ for each $M \in GL_n(\mathbb{R})$ and $x, y \in \mathbb{R}^n$, then it follows that χ preserves the non-degeneracy of bilinear forms, and $\mathcal{Q}_{\mathbb{R}}(\Omega) \cong \mathcal{Q}_{\mathbb{R}}(\Omega')$ if and only if Ω and Ω' belong to the same $GL_n(\mathbb{R})$ -orbit of χ , that is, if and only if $\Omega' = \Omega \circ M$ for some $M \in GL_n(\mathbb{R})$.

Therefore, since the action of χ corresponds to a change of generators in $\mathcal{A}_{\mathbb{R}}(\Omega)$, it follows that $\mathcal{A}_{\mathbb{R}}(\Omega)$ only depends by the orbit of Ω under χ . So, we may define the *moduli space* $\mathcal{M}(\mathcal{Q}_{\mathbb{R}}(\Omega))$ of $\mathcal{Q}_{\mathbb{R}}(\Omega)$, to be the space of all $GL_n(\mathbb{R})$ -orbits of χ .

Furthermore, taking into account what has been said above about $\mathcal{A}_{\mathbb{R}}(\Omega)$ in \mathbb{R}^n , by Proposition 20 of [12], Appendix 2, follows that there is a unique algebra homomorphism $\Delta_t : \mathcal{Q}_{\mathbb{R}}(\Omega) \rightarrow \mathcal{A}_{\mathbb{R}}(\Omega) \otimes \mathcal{Q}_{\mathbb{R}}(\Omega)$ such that $\Delta_t(x^j) = t_i^j \otimes x^i$ for all $j = 1, \dots, n$, endowing $\mathcal{Q}_{\mathbb{R}}(\Omega)$ of a $\mathcal{A}_{\mathbb{R}}(\Omega)$ -comodule structure. Hence, the quantum group of Ω coacts on the quantum space corresponding to $\mathcal{Q}_{\mathbb{R}}(\Omega)$, that is $\mathcal{Q}_{\mathbb{R}}(\Omega)$ corresponds to the natural quantum space for the coaction of $\mathcal{A}_{\mathbb{R}}(\Omega)$.

Come back to the case $n = 4$, in [1] the Hopf algebra $\mathcal{A}_{\mathbb{R}}(\Omega)$ is also endowed with a particular quasi-triangular structure through a R -matrix, say $\mathcal{R} : \mathbb{R}^4 \otimes \mathbb{R}^4 \rightarrow \mathbb{R}^4 \otimes \mathbb{R}^4$, given by $\mathcal{R}_a = \tau + a(\Omega^{-1})^t \otimes \Omega$, where $a \in \mathbb{R} \setminus \{0\}$ and τ is the flip map.

Indeed, for $a \neq 0$, we have the following homogeneous defining relations of $\mathcal{A}_{\mathbb{R}}(\Omega)$:

$$\mathcal{R}_{k_1 k_2}^{i_1 i_2} t_{j_1}^{k_1} t_{j_2}^{k_2} = t_{k_1}^{i_1} t_{k_2}^{i_2} \mathcal{R}_{j_1 j_2}^{k_1 k_2}, \quad i_l, j_l = 0, 1, 2, 3, \quad l = 1, 2,$$

so that such \mathcal{R} is a R -matrix because it satisfy the following, well-known Yang-Baxter equation $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$ when and only when $a \in \mathbb{R} \setminus \{0\}$ verify the braid relation $a(1 + a\Omega^{ij}\Omega_{ij} + a^2) = 0$, equivalent (since $a \neq 0$) to $a + a^{-1} + \Omega^{ij}\Omega_{ij} = a + a^{-1} + tr(\Omega^{-1}\Omega^t) = 0$.

Thus, we have a R -matrix for $\mathcal{A}_{\mathbb{R}}(\Omega)$, given by $\mathcal{R} = \tau + a(\Omega^{-1})^t \otimes \Omega$ with $a \in \mathbb{R} \setminus \{0\}$ such that $a + a^{-1} + \Omega^{ij}\Omega_{ij} = 0$.

In [14], it is proved that the representation category of $\mathcal{A}_{\mathbb{R}}(\Omega)$ (in \mathbb{R}^n , $n \geq 2$) is monoidally equivalent to the representation category of the quantum group $\mathcal{O}_a(SL_2(\mathbb{R}))$ of functions over $SL_2(\mathbb{R})$, if $a \in \mathbb{R} \setminus \{0\}$ verify the above braid relation, so that $Comod(\mathcal{A}_{\mathbb{R}}(\Omega)) \cong^{\otimes} Comod(\mathcal{O}_a(SL_2(\mathbb{R})))$.

Moreover, in the § 5. of [14] it is also presented the following isomorphic

classification of the Hopf algebra $\mathcal{A}_{\mathbb{R}}(\Omega)$: if Ω and Ω' are non-degenerate bilinear forms respectively in \mathbb{R}^n and \mathbb{R}^m with $n, m \geq 2$, then $\mathcal{A}_{\mathbb{R}}(\Omega)$ and $\mathcal{A}_{\mathbb{R}}(\Omega')$ are isomorphic if and only if $m = n$ and there exists $M \in GL_n(\mathbb{R})$ such that $\Omega' = M^t \Omega M$.

Then, in the § 6 of [14], the Author determines the possible Hopf $*$ -algebra structures and CQG (compact quantum group) algebra structures on $\mathcal{A}_{\mathbb{C}}(\Omega)$ (that is, in the complex case).

Following the results of [14], T. Aubriot, in [15], studies the possible Galois and bi-Galois objects over $\mathcal{A}_{\mathbb{R}}(\Omega)$.

At last, the paper [1] finishes with some remarks; in particular, the Authors notices that, in dimension $n \geq 3$, there is no Ω such that $\mathcal{A}_{\mathbb{R}}(\Omega)$ be commutative, that is to say, a such Hopf algebra is necessarily non-commutative.

On the other hand, we remember that in \mathbb{R}^4 may be establish a standard canonical complex structure as follows.

Respect to the canonical base of \mathbb{R}^4 , if $J_0 \in \text{End}(\mathbb{R}^4)$ is defined putting $J_0(e_j) = e_{n+j}$ for $1 \leq j \leq 2$ and $J_0(e_j) = -e_{j-n}$ for $3 \leq j \leq 4$, then it follows that such a J_0 is a complex structure⁷ on \mathbb{R}^4 , and if $\mathbb{R}_{\mathbb{C}}^4(J_0)$ is the resulting linear complex space structure induced by J_0 on \mathbb{R}^4 , then we have the canonical isomorphism $\mathbb{R}_{\mathbb{C}}^4(J_0) \cong \mathbb{C}^2$.

From here, it is possible to construct the following faithful representation $\rho : M^{(2,2)}(\mathbb{C}) \rightarrow M^{(4,4)}(\mathbb{R})$ defined by

$$\rho(A + iB) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

that it is a \mathbb{R} -algebra monomorphism such that $\rho(iH) = J_0 \rho(H)$ for any $H \in M^{(2,2)}(\mathbb{C})$, extending the usual immersion⁸ $GL_2(\mathbb{C}) \hookrightarrow GL_4(\mathbb{R})$.

Hence, if $\Omega_{ij} \in M^{(4,4)}(\mathbb{R})$ of (2'), is such that $\Omega_{ij} \in \rho(M^{(2,2)}(\mathbb{C}))$, let $\tilde{\Omega}_{ij} = \rho^{-1}(\Omega_{ij}) \in M^{(2,2)}(\mathbb{C})$; whence, we may identifies Ω_{ij} with $\tilde{\Omega}_{ij}$, that it is a non-degenerate (if such is Ω_{ij}) bilinear form of \mathbb{C}^2 .

Therefore, if $\mathcal{A}_{\mathbb{C}}(\tilde{\Omega})$ is the Hopf algebra associated to $\tilde{\Omega}$, then it is immediate to prove that $\mathcal{A}_{\mathbb{R}}(\Omega) \cong \mathcal{A}_{\mathbb{C}}(\tilde{\Omega})$.

In [1], § 6., there is a complete classification of the moduli space of $\tilde{\Omega}$, according to the rank of $\tilde{\Omega}$. Precisely

- if $rk \tilde{\Omega} = 0$, then there is only one orbit of which one representative el-

⁷Since $J_0^2 = -id_{\mathbb{R}^4}$.

⁸For any $n \geq 2$, we remember that there exists a well-known immersion $GL_n(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{R})$.

ement is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, this case corresponding to $SL_2(\mathbb{C})$ with R -matrix the identity R_0 of $\mathbb{C}^2 \otimes \mathbb{C}^2$;

- if $rk \tilde{\Omega} = 1$, then there is only one orbit of which one representative element is $\begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}$ with $\lambda \neq 0$ (these are all equivalent among them), this case corresponding to the so called *Manin's jordanian* (that it is a special quantum deformation of $SL_2(\mathbb{C})$; see [3]), with equivalent R -matrices R_λ such that $\lim_{\lambda \rightarrow 0} R_\lambda = R_0$;
- if $rk \tilde{\Omega} = 2$, then there are many orbits, each represented by $\tilde{\Omega}_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$ for every $q \in \mathbb{C} \setminus \{0, 1\}$, with $\mathcal{A}_{\mathbb{C}}(\tilde{\Omega}_q) \cong SL_{2,q}(\mathbb{C})$, R -matrix corresponding to that of $M_{2,q}(\mathbb{C})$ (quantum deformation of $M^{(2,2)}(\mathbb{C})$; see [3]), and $\mathcal{Q}_{\mathbb{C}}(\tilde{\Omega}_q)$ corresponding to the *Manin plane* (that it is the natural quantum space for the coaction of $SL_{2,q}(\mathbb{C})$).

The considerations of this paper, may have physical interpretations in view of the possible physical meaning of Ω (and of $\tilde{\Omega}$, when $\tilde{\Omega}$ exists).

References.

- [1] M. Dubois-Violette, G. Launer, "The quantum group of a non-degenerate bilinear form", Physics Letters B, 245(2) (1990) 175-177.
- [2] S. Majid, Foundations of quantum group theory, Cambridge University Press, Cambridge, 1995.
- [3] Yu.I. Manin, Quantum groups and Non-Commutative Geometry, Publications du CRM de l'Université de Montréal, Montréal, 1988.
- [4] C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation, W.H. Freeman and Company, San Francisco, 1973.
- [5] R.K. Sachs, H. Wu, General Relativity for Mathematicians, Springer-Verlag, New York, 1977.
- [6] J. Stewart, Advanced General Relativity, Cambridge University Press, Cambridge, 1991.
- [7] R.M. Wald, General Relativity, University of Chicago Press, Chicago,

1984.

[8] M. Dubois-Violette, "On the theory of quantum groups", Letters in Mathematical Physics, 19 (1990) 121-126.

[9] M. Francaviglia, Relativistic Theories, Quaderni del GNFM-CNR, Firenze, 1988.

[10] L. Nobili, Astrofisica Relativistica, CLEUP Editrice, Padova, 2003.

[11] J. Madore, An Introduction to Noncommutative Geometry and its Physical Applications, LMS 206, Cambridge University Press, Cambridge, 1998.

[12] M. Dubois-Violette, "Multilinear Forms and Graded Algebras", Journal of Algebra, 317 (2007) 198-225.

[13] M. Dubois-Violette, "Graded algebras and multilinear forms", C.R. Acad. Sci. Paris, Ser. I, 341 (2005) 719-724.

[14] J. Bichon, "The representation category of the quantum group of a non-degenerate bilinear form", Comm. Alg., 31 (2003) 4831-4851.

[15] T. Aubriot, "On the classification of Galois objects over the quantum group of a nondegenerate bilinear form", Man. Math., 122 (2007) 119-135.

[16] D. Lovelock, "The four-dimensionality of space and the Einstein's tensor", Journal of Mathematical Physics, 13 (6) (1972) 874-876.

[17] T. Frankel, Gravitational Curvature, W.H. Freeman and Comp., San Francisco, 1979.